

Nonreciprocal wave propagation through open, discrete nonlinear Schrödinger dimers

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Summary. We consider asymmetric (nonreciprocal) wave transmission through a layered nonlinear, non mirror-symmetric system described by the one-dimensional Discrete Nonlinear Schrödinger equation with spatially varying coefficients embedded in an otherwise linear lattice. Focusing on the simplest case of two nonlinear sites (the dimer), we compute exact scattering solutions such that waves with the same frequency and incident amplitude impinging from left and right directions have different transmission coefficients. The stability of some particular solutions is addressed. We show that oscillatory instability may lead to the formation of stable extended states coexisting with a nonlinear defect mode oscillating at a different frequency. Numerical simulations of wave packet scattering are presented. Gaussian wave packets with the same amplitude arriving from opposite directions on the dimer are indeed transmitted very differently. Moreover, asymmetric transmission is sensitively dependent on the input parameters, akin to the case of chaotic scattering.

1 Introduction

The possibility to control energy and/or mass flows using nonlinear features of physical systems is a fascinating issue both from the point of view of basic science as well as from the applied one. In the context of wave propagation through nonlinear media, the simplest form of control would be to devise a “wave diode” in which electromagnetic or elastic waves are transmitted differently along two opposite propagation directions.

In a linear, time-reversal symmetric system this possibility is forbidden by the reciprocity theorem. As stated by Lord Rayleigh in his treatise *The theory of sound* [1]:

Let A and B be two points [...] between which are situated obstacles of any kind. Then a sound originating at A is perceived at B with the same intensity as that with which an equal sound originating at B would be perceived at A. In acoustics [...] in consequence of the not insignificant value

of the wavelength in comparison with the dimension of ordinary obstacles the reciprocal relation is of considerable interest.

To achieve the desired effect one must thus violate the hypothesis of the theorem. In linear systems, a standard way is to break the time-reversal symmetry by applying a magnetic field as done, for instance, in the case of optical isolators. An entirely alternative possibility is instead to consider *nonlinear* media. At least in principle, this option would offer a whole new range of possibilities of propagation control based on intrinsic material properties rather than an external field.

Asymmetric wave propagation induced by nonlinearity arises in several different domains. Among the first examples discussed in the literature is the asymmetric phonon transmission through a nonlinear interface layer between two very dissimilar crystals [2]. In the field of nonlinear optics a relatively vast number of approaches exist. A so-called all-optical diode has been proposed first in Ref. [3, 4] and later on realized experimentally [5]. There are also proposals to employ left-handed metamaterials [6], quasiperiodic systems [7], coupled nonlinear cavities [8] or \mathcal{PT} -symmetric waveguides [9, 10]. Extension to the quantum regime in which few-photon states display a diode effect has been proposed [11].

In the realm of acoustics the possibility of realizing a diode has been demonstrated for nonlinear phononic media [12, 13]. Another promising context is the propagation of acoustic pulses through granular systems. Indeed, experimental studies demonstrated a change of solitary wave reflectivity from the interface of two granular media [14]. More recently, demonstration of rectification of mechanical energy at sonic frequencies in a one-dimensional array of particles has been also reported [15].

Despite the variety of physical contexts, the basic underlying rectification mechanisms rely on nonlinear phenomena as, for instance, second-harmonic generation in photonic [16] or phononic crystals [12], or bifurcations [15]. In those examples the rectification depends on whether some harmonic (or subharmonic) of the fundamental wave is transmitted or not.

A related question is the possibility that the transmitted power at the *same frequency and incident amplitude* would be sensibly different in the two opposite propagation directions. In this Chapter we address the above problem with the Discrete Nonlinear Schrödinger (DNLS) equation [17, 18] with site-dependent coefficients. It has been demonstrated [19] that DNLS equation can be a sensible approximation for the evolution of longitudinal Bloch waves in layered photonic or phononic crystals. Variable coefficients describe different nonlinear properties of each layer and the presence of defects. In the realm of the physics of cold atomic gases, the equation is an approximate semiclassical description of bosons trapped in periodic optical lattices (see e.g. Ref. [20] and references therein for a recent survey). Beyond its relevance in many different physical contexts, the DNLS equation has the big advantage of being among the simplest dynamical systems amenable to a complete theoretical analysis. For our purpose, it is particularly convenient as it allows to solve the scattering problem exactly without the complications of having to deal with wave harmonics [21].

In Sec. 2 we outline the model and show some examples of asymmetric plane-wave solutions. The issue of their stability is briefly addressed in Sec. 3. It is also shown that oscillatory instability may lead to the formation of stable extended states coexisting with a nonlinear defect mode oscillating at a different frequency. In Sec. 4

we report some numerical simulation of wave packet's scattering and illustrate its dependence on initial intensity. Finally, a brief summary of the results is given in Sec. 5.

2 The model

The DNLS equation with spatially varying coefficients, defined on an infinite one-dimensional lattice is given by

$$i\dot{\phi}_n = V_n\phi_n - \phi_{n+1} - \phi_{n-1} + \alpha_n|\phi_n|^2\phi_n, \quad (1)$$

where units have been chosen such as the coupling $C = 1$.

We will assume the usual scattering setup where V_n and α_n are non vanishing only for $1 \leq n \leq N$. The two semi infinite portions ($n < 1$, $n > N$) of the lattice, model two leads where the wave can propagate freely [22]. Let us look for solutions of the associated stationary transmission problem $\phi_n(t) = u_n \exp(-i\mu t)$

$$\mu u_n = V_n u_n - u_{n+1} - u_{n-1} + \alpha_n |u_n|^2 u_n \quad 1 \leq n \leq N \quad (2)$$

of the form

$$u_n = \begin{cases} R_0 e^{ikn} + R e^{-ikn} & n \leq 1 \\ T e^{ikn} & n \geq N \end{cases} \quad (3)$$

where $\mu = -2 \cos k$ and $0 \leq k \leq \pi$ for the wave coming from the left direction; R_0, R and T are the incident, reflected and transmitted amplitudes respectively. The solution sought must be complex in order to carry a non vanishing current $J = 2|T|^2 \sin k$.

To break the mirror symmetry with respect to the center of the nonlinear portion, one must choose at least one of the two sets of coefficients V_n, α_n such that $V_n \neq V_{N-n+1}$, $\alpha_n \neq \alpha_{N-n+1}$. Note that the transmission of the right-incoming wave with the same R_0 and μ is computed by solving the problem with $(V_n, \alpha_n) \rightarrow (V_{N-n+1}, \alpha_{N-n+1})$ (i.e. “flipping the sample”). In the following, we will adopt the convention to label with $-k$ the right-incoming solutions with wave number k . Nonlinearity is essential as for $\alpha_n = 0$ the transmission coefficient is the same for waves coming from the left or right side, independently on V_n due to time-reversal invariance of the underlying equations of motion [23].

The standard way to solve the problem is to introduce the (backward) transfer map [24, 25, 26, 22]

$$u_{n-1} = -v_n + (V_n - \mu + \alpha_n |u_n|^2) u_n, \quad v_{n-1} = u_n. \quad (4)$$

Note that these are complex quantities therefore the map is nominally four-dimensional. However, due to conservation of energy and norm, it can be reduced to a two-dimensional area-preserving map [24, 25, 26, 22] with an additional control parameter (the conserved current J). The solutions are straightforwardly found by iterating (4) from the initial point $u_N = T \exp(ikN)$, $v_N = T \exp(ik(N+1))$ dictated by the boundary conditions of Eq. (3). For fixed T and k , the incident and reflected amplitudes are determined as

$$R_0 = \frac{\exp(-ik)u_0 - v_0}{\exp(-ik) - \exp(ik)}, \quad R = \frac{\exp(ik)u_0 - v_0}{\exp(ik) - \exp(-ik)}$$

and the transmission coefficient is $t(k, |T|^2) = |T|^2/|R_0|^2$. Note that if $(u_0, v_0) = (u_N, v_N)$ (periodic point of the map) then $t = 1$.

For very short chains (oligomers), t can be computed analytically. For instance for the dimer $N = 2$:

$$t = \left| \frac{e^{ik} - e^{-ik}}{1 + (\delta_2 - e^{ik})(e^{ik} - \delta_1)} \right|^2 \quad (5)$$

where

$$\begin{aligned} \delta_1 &= V_2 - \mu + \alpha_2 T^2, \\ \delta_2 &= V_1 - \mu + \alpha_1 T^2 [1 - 2\delta \cos k + \delta_1^2]. \end{aligned}$$

For the trimer $N = 3$

$$t = \left| \frac{e^{ik} - e^{-ik}}{e^{ik} - \delta_1 + (e^{ik} - \delta_3)(1 - \delta_2(\delta_1 - e^{ik}))} \right|^2 \quad (6)$$

where

$$\begin{aligned} \delta_3 &= V_3 - \mu + \alpha_3 |T|^2 \\ \delta_2 &= V_2 - \mu + \alpha_2 |T|^2 |\delta_3 - e^{ik}|^2 \\ \delta_1 &= V_1 - \mu + \alpha_1 |T|^2 [1 + \delta_2 (e^{ik} - \delta_3)]^2. \end{aligned}$$

The formulas apply for $k > 0$ (left-incoming waves); the transmission for right-incoming waves is obtained by exchanging the subscripts 1 and 2.

Two examples of the dependence of t on the input power are shown in Fig. 1. The curves display a multistable behavior and, for strong enough intensities, are sizeably different indicating nonreciprocity. The effect is maximal in the vicinity of the nonlinear resonances that are detuned differently for the $k > 0$ and $k < 0$ cases yielding intervals of input values where multiple solutions exist only for one propagation direction [21]. The lower panel of Fig. 1 shows that even a moderate increase of the number of sites ($N = 3$) dramatically increase the complexity of the curves as expected due to the mixed phase-space of the underlying transfer map [24].

To conclude this section, we mention that an alternative approach would be to use the Green's function formalism previously used to compute the stationary states for an electron moving on a chain with nonlinear impurities [27]. Indeed, for the case of a symmetric nonlinear dimer, resonance phenomena are demonstrated that lead to complete transmission through the dimer [27]. Of course, we expect such an approach to yield the same results when applied to the present case.

3 Stability of scattering solutions

An important issue is the dynamical stability of the solutions. To the best of our knowledge no systematic study of scattering solutions of the type described above has been presented in the literature (see [28] for the case of NLS equation with concentrated nonlinearities continuum case and [29] for an analysis of a related problem, the nonlinear Fano effect).

The linear stability analysis is performed [17, 18] by letting $\phi_n = (u_n + \chi_n) \exp(-i\mu t)$, and linearizing the equation of motion to obtain

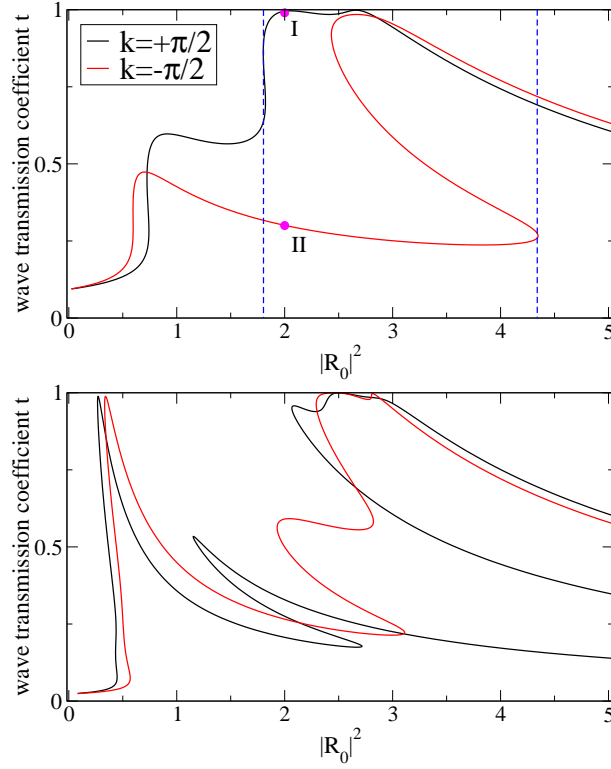


Fig. 1. Transmission coefficients as a function of the input intensity $|R_0|^2$ for $k = \pi/2$, $\alpha_n = 1$. Upper panel: dimer $V_{1,2} = V_0(1 \pm \varepsilon)$. Lower panel: trimer $V_{1,3} = V_0(1 \pm \varepsilon)$, $V_2 = 0$; $V_0 = -2.5 \varepsilon = 0.05$.

$$i\dot{\chi}_n = (V_n - \mu)\chi_n - \chi_{n+1} - \chi_{n-1} + \alpha_n (2|u_n|^2\chi_n + u_n^2\chi_n^*) \quad (7)$$

Note that χ_n is complex. Letting

$$\chi_n = A_n \exp(i\lambda t) + B_n^* \exp(-i\lambda^* t)$$

then λ is solution of the eigenvalue problem

$$\begin{aligned} \lambda A_n &= -(\varepsilon_n - \mu)A_n + A_{n+1} + A_{n-1} - \theta_n B_n \\ \lambda B_n &= +(\varepsilon_n - \mu)B_n - B_{n+1} - B_{n-1} + \theta_n^* A_n \end{aligned} \quad (8)$$

with $\varepsilon_n \equiv V_n + 2\alpha_n|u_n|^2$, $\theta_n \equiv \alpha_n u_n^2$. Note that, at variance with the case of e.g. breather solutions, the solutions are complex, and also the coefficients θ_n in (8) are complex as well. As it is known, the eigenvalues come in quadruplets of the form $\pm\lambda, \pm\lambda^*$. If eigenvalues have a nonzero imaginary part then the solution is unstable. Generally speaking, equilibria of an Hamiltonian system can lose spectral (and therefore linear) stability in two ways: a pair of real eigenvalues can either

(i) merge at the origin and split onto the imaginary axis (saddle-node bifurcation) or (ii) collide at a nonzero point and split off into the complex plane, forming a complex quadruplet (Krein bifurcation). The latter case correspond to an oscillatory instability.

To solve the linear problem (8) exactly one should impose to the solution a definite plane wave form (with complex wave numbers) in the two semiinfinite linear parts of the chain. The matching of such waves through the nonlinear portion reduces the infinite-dimensional problem (8) to an homogeneous linear system of $2N$ equations, whose solvability condition, along with the dispersion relations, yields a set of nonlinear equations for the unknowns. The details of this method (which is technically more involved than a straightforward diagonalization) will be presented elsewhere [30]. Here, we limit to illustrate the stability properties of some representative solutions by a more direct approach, i.e. by solving numerically the eigenvalue problem for a finite truncation $-N_p \leq n \leq N + N_p$ of the chain (M sites with $M = 2N_p + N + 1$), checking that the relevant eigenvalues of the resulting $2M \times 2M$ matrix are not affected by the truncation error.

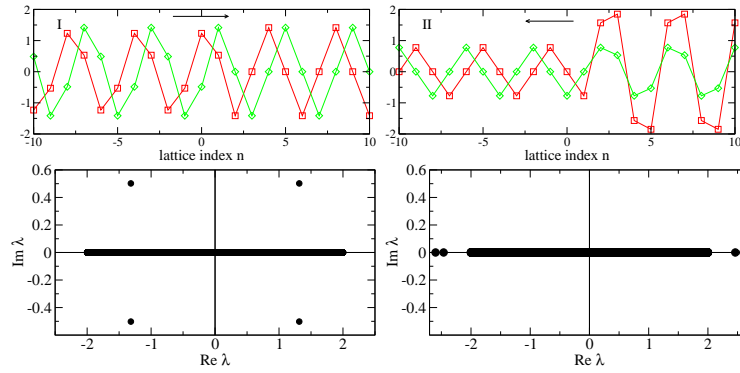


Fig. 2. Upper panels: real (squares) and imaginary (diamonds) parts of the two solutions marked with *I* and *II* in Fig. 1: they correspond to incident waves with the same input $|R_0|^2 = 2$ having transmission coefficients $t = 0.99$ and $t = 0.30$ respectively. Lower panels: the spectrum in the complex plane $N_p = 200$. Isolated eigenvalues for for solution *I* are $\pm 1.316 \pm 0.502i$ for *II*, ± 2.60 and ± 2.46 respectively.

Of course, for small enough nonlinearities/amplitudes the solutions should be stable. Since our main object of interest here is in the large asymmetry effects, we concentrate on the cases of strongly nonlinear waves. Fig. 2 shows two examples of two such solutions corresponding to the same input (marked by dots in the upper panel of Fig.1) along with their eigenvalue spectra. As expected, in both cases there is a continuum component filling densely the interval $[-2, 2]$ on the real axis corresponding to propagation of linear waves⁴. In addition, isolated eigenvalues indicate

⁴ Examining the eigenvalue spectra for different sizes M of the matrix, reveals that this continuum part of the spectra is mostly affected by truncation error. Typically

that the solution I undergoes an oscillatory instability while II is stable. The components of the corresponding eigenvectors are exponentially localized around the nonlinear portion of the chain. This is intuitively clear as only perturbations located there can destabilize the solutions.

For comparison we also integrate the time-dependent DNLS equation setting as initial condition $\phi_n(0) = u_n$ and imposing the boundary conditions $\phi_n(t) = u_n e^{-i\mu t}$ at the two edge sites $n = -N_p$ and $n = N + N_p$ to simulate the infinite system. As seen in Fig.3 the destabilization of solution I occurs by an exponential growth of ϕ_n on the central sites accompanied by an oscillatory behavior with the frequency prescribed by the stability spectrum (see the middle panel of Fig.3). Since this frequency is in the band of linear waves, this process is accompanied by emission of some radiation (traveling peaks in the upper panel of Fig.3) until the amplitude and frequency become large enough leading to a stable localized object (lower panel in Fig.3). This state is reminiscent of a nonlinear defect mode [18]. There is however an important difference as the localized mode is superimposed to a plane wave and that the overall evolution is quasi-periodic.

4 Scattering of wave packets

In this section we illustrate the consequences of the above results on the transmission of wave packets. In a nonlinear system where the superposition principle no longer holds, the connection between the two problems is not trivial. We solved numerically the time-dependent DNLS on a finite lattice $|n| \leq M$ with open boundary conditions, for the case of the dimer discussed in [21]. We take as initial condition a Gaussian wave packet (for $n_0 < 0$)

$$\phi_n(0) = I \exp \left[-\frac{(n - n_0)^2}{w^2} + ik_0 n \right] \quad (9)$$

where $k_0 > 0$ for $n_0 < 0$ (left-incoming packet) and where $k_0 < 0$ for $n_0 > N$ (right-incoming packet). The upper panels of Fig. 4 display the evolution of two packets with the same I and opposite wave number k_0 impinging on the nonlinear dimer. The asymmetry of their propagation is manifest. In both cases, the packets are significantly distorted after scattering, and the emerging envelope may vary wildly. However, the Fourier analysis shows that they remain almost monochromatic at the incident wave number k_0 (lower panels of Fig. 4), with some small background amplitude radiation leaking throughout the lattice.

To quantify the asymmetry of the scattering, and to compare with the above analysis, we measured the wave packet transmission coefficient as the ratio between the transmitted norm at the end t_{fin} of the run divided by the initial one, namely (for $n_0 < 0$)

$$t_p = \frac{\sum_{n>N} |\phi_n(t_{fin})|^2}{\sum_{n<1} |\phi_n(0)|^2}.$$

the numerical eigenvalues have a spurious imaginary part which is of order $1/M$. This is not surprising since the corresponding eigenvectors are extended waves and thus more sensitive to boundary effects.

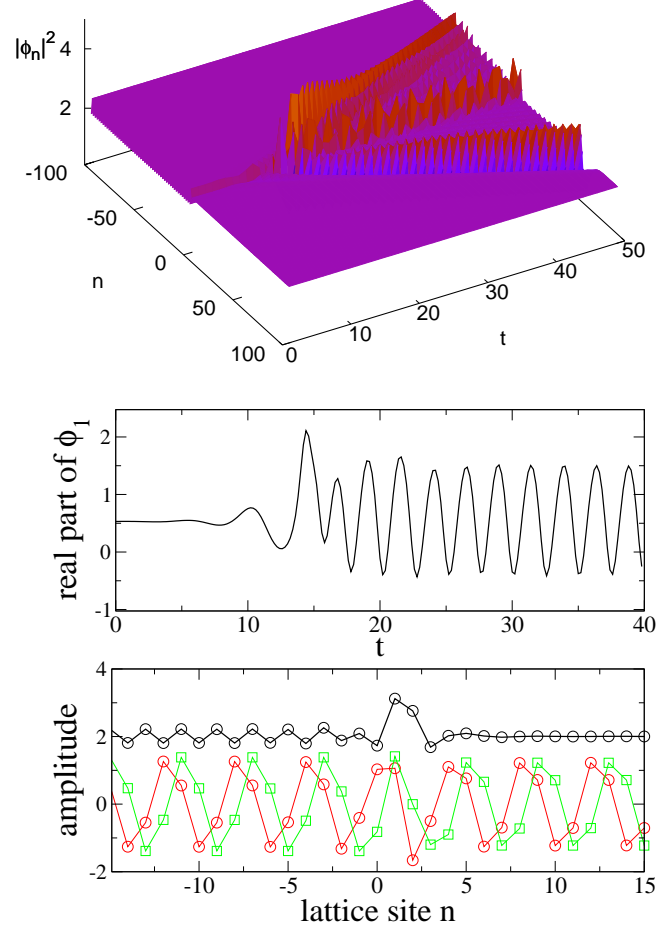


Fig. 3. Upper panel: unstable evolution of the solution I depicted in Fig. 2; numerical integration of DNLS with initial condition corresponding to the stationary orbit with a small perturbation of size 10^{-5} to the site $n = 1$; Middle panel: evolution of $Re\phi_1$: initial oscillations are at an angular frequency close to the imaginary part of the unstable eigenvalue, $Re\lambda = 1.316$. At the later stage, a stable periodic oscillation sets in with frequency increases to 2.5 which is outside of the band of linear waves. Lower panel: snapshots of real (squares) and imaginary (diamonds) parts of central part the chain at $t = 400$. The upper curve is $|\phi_n|^2$ showing the appearance of a localized excitation residing on the dimer.

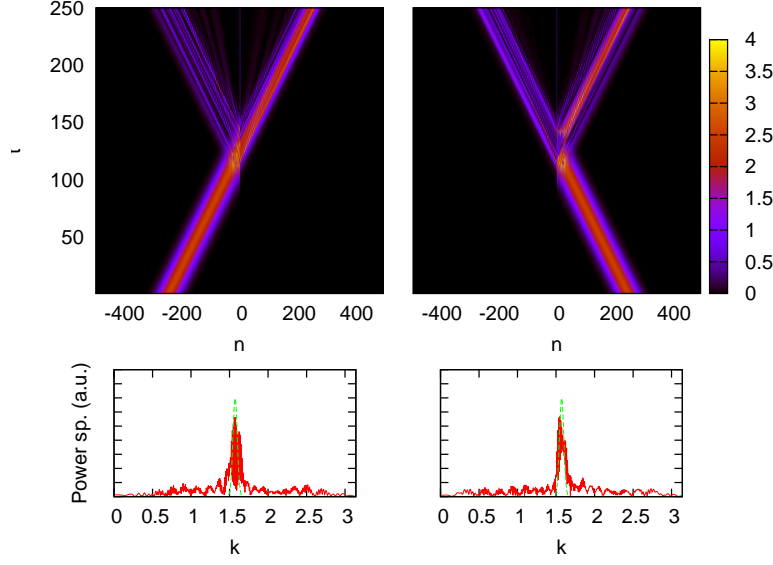


Fig. 4. Numerical simulations of the propagation of Gaussian wave packets, Eq. (9) impinging on a DNLS dimer. Here $V_0 = -2.5$, $|k_0| = 1.57$, $\varepsilon = 0.05$, $M = 500$, $|I|^2 = 2.5$, $w = 50$ and $n_0 = \mp 250$, respectively. Lower panels: Power spectra of the real part of ϕ_n at times $t = 0$ (green dashed line) and $t = 250$ (red solid line).

In Fig. 5 we show the transmission coefficient as a function of the packet intensity $|I|^2$ for two widths of the initial packet. Comparison with Fig. 1 shows that the region of wave amplitudes where the non-reciprocal behavior is maximal (that between the vertical dotted lines) corresponds qualitatively well to the range where packets with comparable intensities are transmitted more asymmetrically.

Besides this, the wider packets show a highly irregular behavior in the sense that there is *sensitive dependence* on initial packets' parameters. This is somehow reminiscent of chaotic scattering (see e.g. [31]). Empirically, we found that this behavior depends very much on the width w : for small w the curves are much smoother. Qualitatively speaking, this can be understood as follows. The dimer can be seen as an integrable, two-degrees of freedom system perturbed by a time-dependent force (the incoming wave packet) and subject to dissipation (the radiation towards the linear leads) [32]. If the packet is very narrow, the duration of the perturbation is short and one may argue that this just determines the initial condition for the dynamical system. The subsequent evolution will be almost regular and a slight change of the forcing will have a little effect. On the contrary, a wider packet will result in a more complex (possibly chaotic) dynamics yielding large changes in the transmission.

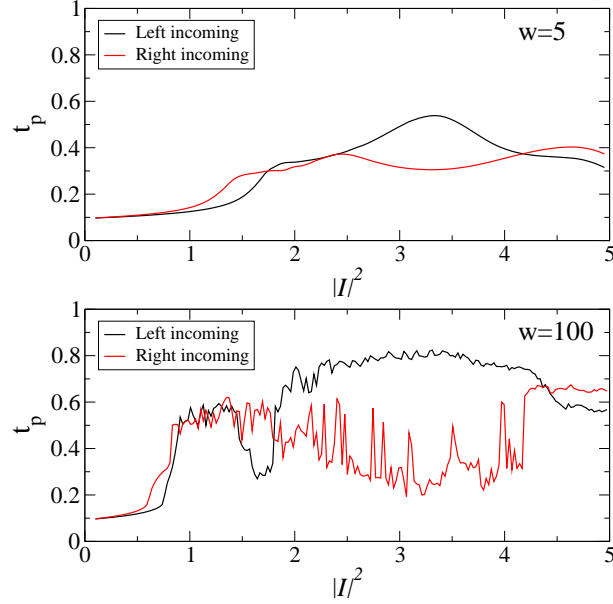


Fig. 5. Dependence of wave packet transmission coefficients on intensity for two different wave packet widths w . Other parameters as in the previous figure.

5 Summary and conclusions

We discussed the scattering problem for a linear Schrödinger chain with an embedded nonlinear, non-mirror symmetric dimer. The simplicity of the model allows to compute analytically a whole family of non reciprocal plane wave solutions, such as left- and right-incoming waves, with the same incident amplitude R_0 and frequency μ , have sizeably different transmission coefficients t .

We then addressed the question of dynamical stability for some specific values of R_0 and μ . Solutions with a large enough t generically undergo oscillatory instabilities (see again Fig. 3). At the initial stage, the development of the instability occurs through growth and oscillation localized around the dimer sites, accompanied by emission of radiation towards infinity. Eventually, a stable extended state emerges coexisting with a nonlinear defect mode oscillating at a different frequency. This numerical finding suggests that the lattice may support exact stable quasiperiodic, nonreciprocal solutions.

Numerical simulations demonstrated that equal Gaussian wave packets impinging on the dimer from the two opposite directions have indeed very different transmission coefficients t_p . The packet amplitudes I for which such non-reciprocal behavior is maximal qualitatively correspond to the range in which extended waves with the same wavenumber are transmitted more asymmetrically. Moreover, t_p is sensitively dependent on the input parameters, like for instance the initial width w and amplitude I (see Fig.5), a feature which is reminiscent of chaotic scattering.

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